Exercise set 2 – Introduction to kinematics

Reminders

Simplified notation of sines and cosines

To simplify the notation, we use:

- $\sin(\theta) = s$
- $\cos(\theta) = c$
- $\sin(\theta_1) = s_1$
- $\cos(\theta_1) = c_1$
- $\sin(\theta_2) = s_2$
- $\cos(\theta_2) = c_2$
- $\cos(\theta_1 + \theta_2) = c_{1+2}$
- $\sin(\theta_1 + \theta_2) = s_{1+2}$

Rotation and translation matrices

Recall that:

- $\mathbf{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ describes the rotation of θ around the origin (in 2D)
- $\mathbf{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$ describes the translation vector $\mathbf{t}$

Trigonometry of the sum of two angles

Recall as well that:

$$\cos(a + b) = \cos(a) \cdot \cos(b) - \sin(a) \cdot \sin(b)$$
$$\sin(a + b) = \sin(a) \cdot \cos(b) + \cos(a) \cdot \sin(b)$$

Sequence of transformations

Finally, the sequence $a \rightarrow b \rightarrow c$ describes the transformation a followed by the transformation b followed by the transformation c.

Exercise 1: 2D rotations around the origin

Calculate the following 2D rotation matrices:

1. $\mathbf{R}(\theta = 0)$.
2. $\mathbf{R}(-\theta)$.
3. $(\mathbf{R}(\theta))^{-1}$.
4. $\mathbf{A} = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$. Also, find θ as a function of θ_1, θ_2 such that $\mathbf{A} = \mathbf{R}(\theta)$.
5. Determine if the following equality is true: $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$.

Exercise 2: Homogeneous transformation matrices in 2D

1. Give the homogeneous matrix corresponding to the pure translation $\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$
2. Give the homogeneous matrix corresponding to the pure rotation $\boldsymbol{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$
3. Give the homogeneous matrix for the following sequence:

$$\mathbf{t} \rightarrow \mathbf{R}(\theta)$$

4. Give the homogeneous matrix for the following sequence:

$$\mathbf{R}(\theta) \rightarrow \mathbf{t}$$

5. Does the homogeneous matrix $\begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$ correspond to the sequence of a translation and a rotation, or to the sequence of a rotation and a translation?
6. Give the homogeneous matrix corresponding to the sequence opposite to that of point 4:

$$\mathbf{R}(\text{-}\theta) \rightarrow \text{-}\mathbf{t}$$

Exercise 3: Sequence of homogeneous matrices

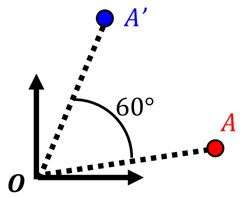
Give the homogeneous matrix for the following sequence of operations:

$$\mathbf{R}(\theta_1) \rightarrow \mathbf{t_1} \rightarrow \mathbf{R}(\theta_2) \rightarrow \mathbf{t_2}$$

Exercise 4: Rotations around an arbitrary point

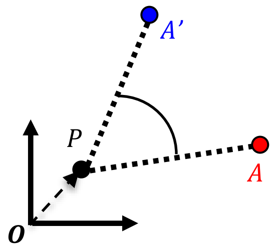
Any combination of translations and rotations in the plane can be expressed as a pure rotation around a point P, called the center of rotation.

1. Is the statement above entirely correct? What about a pure translation?
2. For a combination of translations and rotations:
   (a) Show how the pure center of rotation P can be found graphically. Hint: draw a random vector $\boldsymbol{v}$ in a plane, and also the result of its homogeneous transformation, then find the pure center of rotation P by construction.
   (b) Find the center of rotation analytically, using the formula for a rotation around a point P given by the vector $\mathbf{p}$. Consider the homogeneous matrix of transformation for the combination of translations and rotations as known.
   (c) Find the same center of rotation by reasoning about what happens when the transformation is applied to this center of rotation. Again, consider the homogeneous matrix of transformation for the combination of translations and rotations as known.
3. Give the homogeneous matrix which describes a rotation of $60°$ around the origin.



4. Give the homogeneous matrix which describes a translation of 1 unit in the *x* direction, then a rotation of 60° around the origin.

5. Give the homogeneous matrix which describes a rotation of 60° around the point $P = \begin{pmatrix}1\\1\end{pmatrix}$.



6. An object with two points v_1 and w_1 is moved so that the images of the points are found respectively at positions v_2 and w_2 with respective homogeneous coordinates:

$$v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$v_2 = \begin{pmatrix}\frac{1-\sqrt{3}}{2}\\ \frac{1-\sqrt{3}}{2}\\ 1\end{pmatrix}, \quad w_2 = \begin{pmatrix}\frac{2-\sqrt{3}}{2}\\ \frac{1}{2}\\ 1\end{pmatrix}$$

Give the homogeneous matrix $\mathbf{M}$ which describes this transformation.